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Decay and Stability for Nonlinear Hyperbolic Equations

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This paper deals with the asymptotic stability of the null solution of a semilinear partial differential equation. The La Salle Invariance Principle has been used to obtain the stability results. The first result is given under quite general hypotheses assuming only the precompactness of the orbits and the local existence. In the second part, under some restrictions, sufficient conditions for precompactness of the orbits and decay of solutions are given. An existence and uniqueness theorem is proved in the Appendix. Some examples are given.

INTRODUCTION

Many phenomena in theoretical physics and nonlinear mechanics can be described by means of nonlinear hyperbolic equations and in particular by equations of wave type. This kind of equation arises naturally in scalar field theory (see Berestycki–Lions [4] and Cazenave [8]) and vibrating membrane theory (see the references in Amerio–Prousse [1] and Haraux [16]). This paper is devoted to investigating the asymptotic properties of the solutions and their stability in the Liapunov sense. For this purpose we shall make suitable dissipativness conditions on the nonlinear terms. These hypotheses allow the presence of significance nonlinearities also for the nondissipative part. For instance, it is possible to study in a three-dimensional space equations of cubic type having a cubic dissipative term added.

An important tool in this framework is the use of Liapunov functionals along the lines of the Barbashin–Krasovskii [3] theorem and the La Salle invariance principle (see also Hale [14], Henry [17] and Dafermos [9, 10]). The wave equation with dissipative terms is also studied in Haraux [16] and Webb [28].

Section 2 of this paper is concerned with general asymptotic stability results under the assumptions of the precompactness of the orbits. In the

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third part we shall give some results regarding sufficient conditions to get the precompactness.

In Section 4 estimates obtained in Section 3 are used to prove the decay estimates.

Let us denote by Ω an open bounded subset of $\mathbb{R}^N$ with $\partial\Omega$ sufficiently regular. Let m be an integer number, $m \geqslant 1$, we shall define

$$D(A) = H_0^m(\Omega) \cap H^{2m}(\Omega)$$
$$Au = \sum_{|\alpha|,|\sigma| \leqslant m} (-1)^{|\alpha|} D^\alpha(a_{\alpha,\sigma}(x)\, D^\sigma u) \tag{0.1}$$

for all $u \in D(A)$. Where $\alpha \in N^N$, $|\alpha| = \sum_{i=1}^N \alpha_i$ and

$$D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}.$$

Moreover assume that $a_{\alpha,\sigma}$ are C^∞ real-valued functions such that

$$\sum_{|\alpha|=|\sigma|=m} \xi^\alpha a_{\alpha,\sigma}(x)\, \xi^\sigma \geqslant c_0\, |\xi|^{2m}, \qquad \xi^\alpha = \prod_{i=1}^N \xi^{\alpha_i}; \qquad \xi \in \mathbb{R}^N, x \in \bar{\Omega}. \tag{0.2}$$

In addition we shall assume $A = A^*$, that is, $a_{\alpha,\sigma} = a_{\sigma,\alpha}$. In the following we shall denote by

$$|u|_{0,p} = \left[\int_\Omega |u(x)|^p\, dx\right]^{1/p}, \qquad u \in L^p(\Omega), p \geqslant 1 \tag{0.3}$$

and by

$$|u|_{l,p} = \left[\sum_{|\alpha| \leqslant l} |D^\alpha u|_{0,p}^p\right]^{1/p}, \qquad u \in H^{l,p}(\Omega), p \geqslant 1, l \in N. \tag{0.4}$$

Given two Banach spaces B_0, B_1 both embedded in a topological vector space Z we shall denote by

$$[B_0, B_1]_\sigma, \qquad \sigma \in [0, 1] \tag{0.5}$$

the complex interpolation space between B_0 and B_1 (see Adams [2, Chap. VII-7.64]). Then for all h, $k \in N$, $p \geqslant 1$,

$$[H^{k,p}(\Omega); H^{h,p}(\Omega)]_\sigma = H^{(1-\sigma)k+\sigma h,p}(\Omega). \tag{0.6}$$

Finally, if $A: \mathbb{R}_+ \to \mathbb{R}_+$ is an N-function satisfying the Δ_2 condition near infinity (see Adams [2, Chap. VIII-8.7]), we shall denote by $L_A(\Omega)$ the Orlicz space endowed by the Luxenburg norm

$$|u|_{A,\Omega} = \inf\{\lambda > 0 : |_\Omega A(\lambda^{-1}\, |u(x)|)\, dx \leqslant 1\}. \tag{0.7}$$

We say that B dominates A near infinity if there exists $t_0 > 0$ and $c > 0$ such that for all $t \geqslant t_0$

$$A(t) \leqslant B(ct).$$

In this case $L_B(\Omega)$ is continuously embedded in $L_A(\Omega)$ and

$$|u|_A \leqslant 2c\,|u|_B.$$

1. Position of the Problem

Let us consider a continuous differentiable mapping $f\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We are interested in the following semilinear equation:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(t,x) + (Au)(t,x) = f\left(x, u(t,x), \frac{\partial u}{\partial t}(t,x)\right)\\
&u(0,x) = \varphi(x), \qquad \frac{\partial u}{\partial t}(t,x) = \psi(x), \qquad x \in \Omega,\ t \geqslant 0 \qquad\qquad \text{(H)}\\
&\frac{\partial^k u}{\partial n^k}(t,x) = 0, \qquad x \in \partial\Omega,\ t \geqslant 0,\ k = 0,\ldots, m-1.
\end{aligned}$$

Since A is a positive self-adjoint linear operator on $L^2(\Omega)$ there exists the operator $A^{1/2}$ which is positive and self-adjoint too. Using the strongly continuous semigroup $\{e^{-tA}\colon t \geqslant 0\}$ generated by $-A$ one has the following representation formula

$$A^{1/2}u = \Gamma(1/2)^{-1} \int_0^\infty e^{-tA} t^{-1/2} u\, dt \tag{1.1}$$

for all $u \in D_{1/2} =$ Domain of $A^{1/2}$ (see Friedman [13]). It is usual to write (H) to a system. Denote by

$$\begin{aligned}
&Q = \begin{bmatrix} 0 & 1 \\ -A & 0 \end{bmatrix}, \qquad W(t,x) = \left(u(t,x), \frac{\partial u}{\partial t}(t,x)\right)^t,\\
&F(W) = \left(0, f\left(\cdot, u, \frac{\partial u}{\partial t}\right)\right)^t
\end{aligned} \tag{1.2}$$

then

$$\begin{aligned}
&\frac{d}{dt} W(t) = (QW)(t) + F(W(t))\\
&W(0) = (\varphi, \psi)^t.
\end{aligned} \tag{1.3}$$

It is well known that Q generates a strongly continuous group $\{G(t): t \in \mathbb{R}\}$ on the Hilbert space

$$X = D_{1/2} \oplus L^2(\Omega) \tag{1.4}$$

endowed with the norm

$$|(\varphi, \psi)|_X^2 = |A^{1/2}\varphi|_{0,2}^2 + |\psi|_{0,2}^2. \tag{1.5}$$

The paper of Segal provides existence and uniqueness results of "mild" and classical solutions (see [26]).

2. Asymptotic Stability

In this paragraph we shall make use of the definition of semiflow and orbit (positive semiorbit) and the ideas of stability and asymptotic stability. We refer to the book of Henry [17] and the notes of Salvadori [25]. Moreover recall that

Definition (2.1). Let $\{S(t): t \geqslant 0\}$ be a semiflow on the matric space M. A Liapunov functional for $S(\cdot)$ is a real-valued map V on M such that

$$\dot{V}(x) = \limsup_{t \to 0_+} t^{-1}\{V(S(t)x) - V(x)\} \leqslant 0 \tag{2.1}$$

for all $x \in M$.

Proposition (2.2). *Let $S(\cdot)$ be a semiflow on the Banach space M. Suppose that V is a continuous Liapunov functional on M, $V(0) = 0$ and $V(x) \geqslant c(|x|)$ for all $x \in M$, where c is a continuous strictly increasing function, $c(0) = 0$. Therefore $x = 0$ is Liapunov stable.*

Proposition (2.3) (Invariance Principle). *Assume each bounded orbit is precompact. If there exists a continuous Liapunov functional V such that $V(0) = 0$, let*

$$S = \{x \in M: \dot{V}(x) = 0\} \tag{2.2}$$

and P be the largest positively invariant set in S (that is, the largest subset of S such that $S(t)x \in P$ for all $t \geqslant 0$ provided $x \in P$). Then

$$\lim_{t \to +\infty} \operatorname{dist}(S(t)x, P) = 0. \tag{2.3}$$

Remark. If V verifies the hypotheses of both of the above propositions P is asymptotically stable and attracts the whole space M. We want to apply this theory to our equation (H) using the Liapunov functional

$$V(\varphi, \psi) = |\psi|_{0,2}^2 + |A^{1/2}\varphi|_{0,2}^2 - 2\int_\Omega \int_0^{\varphi(x)} f(x, \sigma, 0)\, d\sigma\, dx. \tag{2.4}$$

The following lemmas will be useful in proving the continuity of the functional V in the norm of X.

LEMMA (2.4). *Assume that $f(x, u, 0)$ is measurable in x and continuous in u. Moreover there exist $p, \gamma \in \mathbb{R}$, $p - 1 \geqslant \gamma$, $p \geqslant 2$, $\gamma \geqslant 1$ and two functions $a_1 \in L^{p/(p-2)}(\Omega)$, $a_2 \in L^{p/(p-\gamma-1)}(\Omega)$, $a_i \geqslant 0$ such that*

$$-(a_1(x) + a_2(x)|u|^{\gamma-1}) \leqslant u^{-1} f(x, u, 0) \leqslant 0, \qquad u \neq 0. \tag{2.5}$$

Then the Nemytskii operator

$$\mathbf{f}(u)(x) = f(x, u(x), 0)$$

is continuous from $L^p(\Omega)$ to $L^{p'}(\Omega)$ and[1]

$$|f(u)|_{0,p'} \leqslant C(p, \Omega)[|a_1|^{0,p/(p-2)}|u|_{0,p} + |a_2|_{0,p/(p-\gamma-1)}|u|_{0,p}^{\gamma}]. \tag{2.6}$$

Proof. For each $\lambda > 0$ denote by

$$u_\lambda(x) = \begin{cases} u(x) & \text{if } |u(x)| \geqslant \lambda \\ 0 & \text{if } |u(x)| < \lambda \end{cases} \tag{2.7}$$

$$\Omega_\lambda = \{x \in \Omega : |u(x)| \geqslant \lambda\}.$$

Therefore

$$\begin{aligned}
&\int_\Omega |f(x, u_\lambda(x), 0)|^{p'}\, dx \\
&= \int_{\Omega_\lambda} \left\{-\frac{f(x, u_\lambda(x), 0)}{u_\lambda(x)}\right\}^{p'} |u_\lambda(x)|^{p'}\, dx \\
&\leqslant C(p)\left[\int_{\Omega_\lambda} a_1^{p'}(x)|u_\lambda(x)|^{p'}\, dx + \int_{\Omega_\lambda} a_2^{p'}(x)|u_\lambda(x)|^{p'\gamma} dx\right] \\
&\leqslant \text{const}[|a_1|_{0,p/(p-2)}^{p'}|u_\lambda|_{0,p}^{p'} + |a_2|_{0,p/(p-\gamma-1)}^{p'}|u_\lambda|_{0,p}^{p'\gamma}].
\end{aligned}$$

Since $|f(x, u_\lambda(x), 0) \leqslant |f(x, u(x), 0)|$, by Lebesgue convergence theorem we get (2.6).

[1] Let us denoted by $p' = p/(p-1)$.

LEMMA (2.5). *Under the above hypotheses denote by*

$$\Phi(u) = \int_\Omega dx \int_0^{u(x)} f(x, \xi, 0)\, d\xi. \tag{2.8}$$

Then Φ is continuously differentiable from $L^p(\Omega)$ to $\mathbb{R}$. *Moreover*

$$\operatorname{grad} \Phi(u) = \mathbf{f}(u). \tag{2.9}$$

Proof. The differentiability can be found in Krasnoselskii [18].

LEMMA (2.6). *If the above hypotheses on f are fulfilled and either*

(i) $N > 2m$ *and* $2 \leqslant p < 2N/(N-2m)$

or

(ii) $N \leqslant 2m$ *and* $2 \leqslant p$

Φ is continuous from $D_{1/2}$ to $\mathbb{R}$.

Proof. By the Sobolev embedding theorem one has

$$|u|_{0,p} \leqslant G_0 |u|_{m,2}, \qquad u \in D(A),\ G_0 > 0 \tag{2.10}$$

provided that $p < 2N/(N-2m)$, $N > 2m$.

Denote by $L_A(\Omega)$ the Orlicz space given by the N-function $A(t) = \exp(t^2) - 1$, thus if $N = 2m$

$$L^p(\Omega) \supset L_A(\Omega) \supset H^{m,2}(\Omega) \tag{2.11}$$

for all $p \geqslant 2$.

Therefore for all $u \in D(A)$ and $N \geqslant 2m$, it follows

$$|u|_{0,p} \leqslant G_1 |u|_{m,2}, \qquad G_1 > 0. \tag{2.10$'$}$$

In the case $N < 2m$ the foregoing inequality follows easily since $H^m(\Omega) \subset C^0(\bar{\Omega})$.

Since A is a positive self-adjoint linear operator one gets

$$D_{1/2} = [D(A); L^2(\Omega)]_{1/2} \tag{2.12}$$

(see Lions–Magenes [20, Proposition 2.1, Chap. I]).

But $D(A)$ is continuously embedded in $H^{2m}(\Omega)$ therefore $[D(A); L^2(\Omega)]_{1/2}$ is continuously embedded in $[H^{2m}(\Omega); L^2(\Omega)]_{1/2} = H^{2m}(\Omega)$. Hence it follows

$$|u|_{m,2} \leqslant \text{const}\, |A^{1/2}u|_{0,2}.$$

Then Φ is continuous from $D_{1/2}$ to $\mathbb{R}$.

THEOREM (2.7). *Suppose that on a certain interval* $[0, T]$, $T > 0$, *there exists a unique classical solution*

$$(u, u_t) \in C^1(0, T; D(Q)) \cap C(0, T; X)$$

for any initial datum $(\varphi, \psi) \in D(Q)$.

Assume that f *is continuous,* $f(\cdot, 0, 0) \equiv 0$, *and the following hypotheses are fulfilled*:

(i) *There exist* $a_1 \in L^{p/(p-2)}(\Omega)$, $a_2 \in L^{p/(p-\gamma-1)}(\Omega)$; $a_1 \geqslant 0$, $a_2 \geqslant 0$, $1 \leqslant \gamma \leqslant p-1$, *such that for all* $x \in \Omega$, $u \neq 0$ *one has*

$$-\{a_1(x) + a_2(x)\,|u|^{\gamma-1}\} \leqslant u^{-1} f(x, u, 0) \leqslant 0 \tag{2.13}$$

where p *has the properties given in Lemma* (2.6).

(ii) *There exists* $b \geqslant 0$ *such that*

$$v^{-1}\{f(x, u, v) - f(x, u, 0)\} \leqslant -b\,|v|^{p-2}, \qquad x \in \Omega,\ v \neq 0. \tag{2.14}$$

Then $(0, 0)$ *is a stable equilibrium point of* X.

Moreover if each bounded orbit is precompact in the X *topology the global existence of the solutions holds. If* $b > 0$ *and* 0 *is in the resolvent set of* A *then* $(0, 0)$ *is asymptotically stable in* X. *Finally, one has*:

(a) $\lim_{t \to +\infty} |A^{1/2} u(t)|_{0,2} = 0$, $\lim_{t \to +\infty} |u_t(t)|_{0,2} = 0$. (2.15)

(b) *If* $N < 2m$, $u(t) \in C^0(\bar{\Omega})$ *for all* $t \geqslant 0$ *and* $\lim_{t \to +\infty} |u(t)|_{0,\infty} = 0$. (2.16)

(c) *If* $N > 2m$, *let* $p \in [2, 2N/(N-2m)]$, *then* $\lim_{t \to +\infty} |u(t)|_{0,p} = 0$. (2.16′)

(d) *If* $N = 2m$, *let* $L_A(\Omega)$ *be the Orlicz space given by* $A(t) = -1 + \exp(t^2)$, *then* $\lim_{t \to +\infty} |u(t)|_{0,p} = \lim_{t \to +\infty} |u(t)|_{L_A} = 0$ *for all* $p \geqslant 2$. (2.17)

Proof. Let us consider the functional V stated in (2.4)

$$V(\varphi, \psi) = |\psi|_{0,2}^2 + |A^{1/2}\varphi|_{0,2}^2 - 2\Phi(\varphi).$$

For all $\lambda > 0$, $\varphi \in C_0^\infty(\Omega)$, let $\Omega_\lambda = \{x \in \Omega : |\varphi(x)| \geqslant \lambda\}$, then

$$\begin{aligned} -2\Phi(\varphi_\lambda) &= -2 \int_{\Omega_\lambda} \int_0^1 \phi_\lambda(x)\, f(x, \sigma\phi_\lambda(x), 0)\, d\sigma\, dx \\ &= -2 \int_{\Omega_\lambda} \phi^2(x) \int_0^1 \sigma\{(\sigma\phi_\lambda(x))^{-1} f(x, \sigma\phi_\lambda(x), 0)\}\, d\sigma\, dx \geqslant 0. \end{aligned}$$

Using the Lebesgue convergence theorem it follows

$$-2\Phi_{\Omega}(\varphi) \geqslant 0. \tag{2.18}$$

Therefore for all $\varphi \in D_{1/2}$ and $\psi \in L^2(\Omega)$

$$V(\varphi, \psi) \geqslant |A^{1/2}\varphi|_{0,2}^2 + |\psi|_{0,2}^2. \tag{2.19}$$

Moreover by Lemma (2.6) V is continuous from $D_{1/2} \oplus L^2(\Omega)$ to $\mathbb{R}$. Now we have to investigate the functional $\dot{V}(\varphi, \psi)$. One has

$$\begin{aligned}\dot{V}(\varphi, \psi) &= -2 \int_{\Omega} \psi(x) f(x, \varphi(x), 0)\, dx + 2 \int_{\Omega} f(x, \varphi(x), \psi(x))\, \psi(x)\, dx \\ &\leqslant -2b \int_{\Omega} \psi^p(x)\, dx.\end{aligned} \tag{2.20}$$

Then for all $(\varphi, \psi) \in D(Q)$, $\dot{V}(\varphi, \psi) \leqslant 0$. Hence $(0, 0)$ is stable. Since the map $t \in \mathbb{R} \to V(u(t), v(t))$ is nonincreasing it follows

$$V(u(t), v(t)) \leqslant V(\varphi, \psi)$$

as long as the solutions exist. The above inequality can be seen as an "a priori estimate" for the solution. Since we assumed the precompactness of the orbits a standard continuation argument provides the global existence of the solutions. By (2.20) one has

$$S = \{(\varphi, \psi) \in D(Q) \colon \dot{V}(\varphi, \psi) = 0\} = \{(\varphi, \psi) \colon \psi = 0\}. \tag{2.21}$$

We shall prove now that the largest invariant set $P \subseteq S$ is $(0, 0)$. Denote by $(\hat{u}(t), \hat{v}(t))$ the solution having initial datum $(\hat{\varphi}, 0) \in S$. If $\hat{\varphi} \neq 0$ we will prove the solution gets outside S for some $\bar{t} > 0$. Indeed one has

$$\begin{aligned}&\limsup_{t \to 0+} \int_{\Omega} \frac{\partial}{\partial t} \hat{v}(t, x)\, \hat{u}(t, x)\, dx \\ &\quad = -\lim_{t \to 0+} |A^{1/2}\hat{u}(t)|_{0,2}^2 \\ &\qquad + \lim_{t \to 0+} \int_{\Omega} \hat{u}(t, x) f(x, \hat{u}(t, x), \hat{v}(t, x))\, dx = -|A^{1/2}\hat{\phi}|_{0,2}^2.\end{aligned} \tag{2.22}$$

Therefore, since

$$\begin{aligned}&-|\hat{\phi}|_{0,2} \left(\limsup_{t \to 0+} \left| \frac{\partial}{\partial t} \hat{v}(t) \right|_{0,2} \right) \\ &\quad \leqslant \lim_{t \to 0+} \int_{\Omega} \frac{\partial}{\partial t} \hat{v}(t, x)\, \hat{u}(t, x)\, dx = -|A^{1/2}\hat{\varphi}|_{0,2}^2\end{aligned} \tag{2.23}$$

it follows

$$\liminf_{t\to 0+} \left| \frac{\partial}{\partial t} v\hat{}(t) \right|_{0,2} \geqslant \frac{1}{\|A^{-1/2}\|} |A^{1/2}\varphi\hat{}|_{0,2}. \tag{2.24}$$

If $(\varphi\hat{}, 0)$ is in a positively invariant subset of S, one has $v\hat{}(t) \equiv 0$, for all $t \geqslant 0$. Then $(\partial/\partial t)\, v\hat{}(t) \equiv 0$ which contradicts (2.24) if $\varphi\hat{} \neq 0$ (since 0 is in the resolvent set of A).

Remark. When $N < 2m$ the hypothesis (2.13) can be weakened. We can assume

$$u^{-1}f(x, u, 0) \leqslant 0, \qquad u \neq 0,\ x \in \Omega. \tag{2.13$'$}$$

Indeed in this case one has the continuity of Φ with respect to the $D_{1/2}$ norm with no power limitation on the growth of f. Let $\{\varphi_n\}$ be a sequence in $D_{1/2}$ converging to $\varphi \in D_{1/2}$ in the $A^{1/2}$ graph norm, since $C^0(\bar{\Omega}) \supset D_{1/2}$ (continuous embedding) then $\varphi_n \to \varphi$ uniformly in $\bar{\Omega}$ and

$$\Phi(\varphi_n) \to \Phi(\varphi) \qquad \text{as} \quad n \to +\infty. \tag{2.25}$$

The above results can be used to investigate the decay and the boundedness of the Faedo–Galerkin approximations.

Let us consider indeed an orthonormal basis for $L^2(\Omega)$, say, $\{e_k\} \subset D(A)$. Define

$$u_m(t) = \sum_{j=1}^{m} g_{j,m}(t)\, e_j \tag{2.26}$$

where $g_{jm}(t)$ are the solutions of the ordinary differential system

$$\begin{aligned} &u_m'' + Au_m = f(x, u_m, u_m'), \qquad u' = \frac{\partial u}{\partial t} \\ &u_m(0) = \sum_{j=1}^{m} \langle \varphi, e_j \rangle\, e_j \\ &u_m'(0) = \sum_{j=1}^{m} \langle \psi, e_j \rangle\, e_j. \end{aligned} \tag{2.27}$$

PROPOSITION (2.8). *If the above hypotheses are fulfilled one has*

$$\lim_{t\to+\infty} |A^{1/2}u_m(t)|_{0,2} = 0, \qquad \lim_{t\to+\infty} |u_m'(t)|_{0,2} = 0 \tag{2.28}$$

and there exists $C > 0$ (*independent from* m) *such that for all* $t \geqslant 0$

$$|A^{1/2}u_m(t)|_{0,2} \leqslant C, \qquad |u_m'(t)|_{0,2} \leqslant C. \tag{2.29}$$

Proof. Let us consider the above Liapunov functional; if we compute $\dot{V}$ along the trajectories of (2.27) it follows

$$\dot{V}_{(2-27)} \leqslant -|u'_m(t)|^2_{0,2}. \tag{2.30}$$

Then using the same arguments of Theorem (2.7) it follows (2.28). By (2.30) we obtain

$$V(u_m(t), u'_m(t)) \leqslant V(u_m(0), u'_m(0)). \tag{2.31}$$

But $V(u_m(0), u'_m(0)) \to V(\varphi, \psi)$ since V is continuous, then (2.29) holds.

3. Precompactness of the Orbits

In this section we assume the local existence and the uniqueness of classical solutions to (H). That is, for all initial data in $D(Q)$ and for all $T > 0$ there exists a unique solution $u \in H^2(0, T; L^2) \cap C^1(0, T; D_{1/2}) \cap C(0, T; D(A))$ and the Galerkin approximations converge in $L^2(0, T; X)$.[2] In the Appendix we show that this follows from our estimates on the Galerkin approximations.

We are interested here in giving sufficient conditions in order to have the precompactness of the orbits in the norm of X. The estimates obtained here are strictly related with those used for the existence of almost periodic solutions (see Amerio–Prouse [1] and Haraux [16]). Also in this case, the idea is to get estimates on the higher-order derivatives, differentiating the equation with respect to t. In this case if we try to apply the "processes theory" (see Dafermos [9, 10]) to this new equation, we have to require $\partial f/\partial v$ to be bounded, that is, a sublinear dissipative term. This case has been studied, with different techniques, by Webb [28] and Narazaki [23]. If we have superlinearity in the dissipation without nonlinear term in u the estimates are given in Amerio–Prouse [1] and Haraux [15, 16].

Our goal has been to find estimates of this kind in presence of superlinear terms both in u and u'.

We shall make use of a simple inequality. If η is a nonnegative real-valued function and

$$\eta(t) \leqslant c_1 \sqrt{\eta(t)} + c_2, \qquad c_1, c_2 \geqslant 0$$

thus $\eta(t) \leqslant \text{const}$.

[2] If V is a Banach space, $C(0, T; V)$ is the space of continuous maps from $[0, T]$ to V endowed with the sup norm.

PROPOSITION (3.1). *Assume that $p = 2N/(N-2m)$ if $N > 2m$ or $2 \leqslant p$ if $N \leqslant 2m$. Moreover let*

$$f(x, u, v) = h(x, u) + g(x, v), \qquad x \in \bar{\Omega};\ u, v \in \mathbb{R}. \tag{3.1}$$

In addition let f be continuously differentiable and

$$(\partial h/\partial u) \leqslant 0 \qquad (\partial g/\partial v) \leqslant 0. \tag{3.2}$$

If the following growth conditions are fulfilled[3]

$$\text{(i)}\quad |h(x, u)| \leqslant C_1(1 + |u|^{p/2}),\ |(\partial h/\partial u)(x, u)| \leqslant C_2(1 + |u|^{q/2}),\quad C_1, C_2 > 0, \tag{3.3}$$

$$\begin{aligned} &\text{(ii)}\quad \text{there exists } \beta > 0 \text{ such that } q(1 + (1/\beta)) \leqslant p \text{ and } q \leqslant p - 2\\ &-C_3(1 + |v|^{-1+p/2}) \leqslant v^{-1} g(x, v) \leqslant -C_4(1 + |v|^{\beta}), \qquad v \neq 0\\ &(\partial g/\partial v)(x, v) \leqslant -C_4(1 + |v|^{\beta}), \qquad C_3, C_4 > 0 \end{aligned} \tag{3.4}$$

then the Galerkin approximations verify the following estimates. There exists $K > 0$ independent from t and m such that

$$|u_m''(t)|_{0,2} \leqslant K, \qquad |A^{1/2} u_m'(t)|_{0,2} \leqslant K, \qquad |A u_m(t)|_{0,2} \leqslant K. \tag{3.5}$$

Proof. Denote by $Y_m(t, \cdot) = (u_m(t, \cdot),\ u_m'(t, \cdot))^{\mathrm{t}}$ (t means transposed). Then it follows

$$Y_m''(t, x) + (A Y_m)(t, x) = G(x, Y_m(t, x))\, Y_m'(t, x) + F(x, Y_m(t, x)) \tag{3.6}$$

where we set

$$F(x, u, v) = (f(x, u, v), 0)^{\mathrm{t}}, \qquad F_0(x, u, v) = (f(x, u, 0), 0)^{\mathrm{t}} \tag{3.7}$$

$$G(x, u, v) = \left[\begin{array}{c|c} \bigcirc & \bigcirc \\ \hline \dfrac{\partial h}{\partial u}(x, u) & \dfrac{\partial q}{\partial v}(x, v) \end{array}\right]. \tag{3.8}$$

Let us define the energy function

$$E_m(t) = |Y_m'(t)|_{0,2}^2 + |A^{1/2} Y_m(t)|_{0,2}^2 - 2\Phi(u_m(t)). \tag{3.9}$$

[3] Actually the hypothesis (3.4) can be weakened (see Remark A.3)).

Our goal is to show that $E_m(t)$ is bounded independently from m. It follows

$$\begin{aligned}\frac{1}{2} E'_m(t) &= \langle Y'_m(t), F(\cdot, Y_m(t)) + G(\cdot, Y_m(t))\, Y'_m(t) - F_0(Y_m(t))\rangle \\
&= \langle u'_m(t), f(\cdot, u_m(t), u'_m(t)) - f(\cdot, u_m(t), 0)\rangle \qquad (3.10)\\
&\quad + \left\langle \frac{\partial h}{\partial u} (\cdot, u_m(t))\, u'_m(t), u''_m(t)\right\rangle \\
&\quad + \left\langle \frac{\partial g}{\partial v} (\cdot, u'_m(t))\, u''_m(t), u''_m(t)\right\rangle \\
&\leqslant - c_4 \int_\Omega (|u'_m(t,x)|^2 + |u''_m(t,x)|^2)(|u'_m(t,x)|^\beta)\, dx \\
&\quad + \int_\Omega \left| \frac{\partial h}{\partial u} (x, u_m(t,x))\right| |u'_m(t,x)|\, |u''_m(t,x)|\, dx \\
&= \frac{1}{2} \int_\Omega Q(t,x)\, dx \leqslant \frac{1}{2} \int_\Omega \left(\left| \frac{\partial h}{\partial u} (x, u_m(t,x))\right| \right. \\
&\quad \left. - 2c_4(1 + |u'_m(t,x)|^\beta)\right) (|u'_m(t,x)|^2 + |u''_m(t,x)|^2)\, dx.\end{aligned}$$

Denote by

$$\begin{aligned}&\Omega_+(t) = \left\{ x \in \Omega : \left| \frac{\partial h}{\partial u} (x, u_m(t,x))\right| \geqslant 2c_4(|u'_m(t,x)|^\beta + 1)\right\} \\
&\Omega_-(1) = \Omega \backslash \Omega_+(t)\end{aligned} \qquad (3.11)$$

and if $n \in N$

$$\begin{aligned}&\Omega^{(n)}_-(t) = \left\{ x \in \Omega_-(t) : \left| \frac{\partial h}{\partial u} (x, u_m(t,x))\right| \geqslant 2c_4(|u'_m(t,x)|^\beta + 1) - \frac{c_4}{2^n}\right\} \\
&\tilde{\Omega}_-(t) = \Omega \backslash (\Omega_+(t) \cup \Omega^{(0)}_-(t)).\end{aligned} \qquad (3.11')$$

Therefore

$$\begin{aligned}\frac{1}{2} E'_m(t) \leqslant \frac{1}{2} \int_{\Omega_+(t)} Q(t,x)\, dx &+ \sum_{n=0}^{\infty} \frac{1}{2} \int_{\Omega^{(n)}_-(t) \backslash \Omega^{(n+1)}_-(t)} Q(t,x)\, dx \\
&+ \frac{1}{2} \int_{\tilde{\Omega}_-(t)} Q(t,x)\, dx.\end{aligned} \qquad (3.12)$$

For all $x \in \Omega_+(t)$ one has

$$2c_4 |u_m'(t,x)|^\beta \leqslant \left| \frac{\partial h}{\partial u}(x, u_m(t,x)) \right| \leqslant c_2(1 + |u_m(t,x)|^{q/2}). \tag{3.13}$$

Then

$$|u_m'(t,x)| \leqslant c(1 + |u_m(t,x)|^{q/2\beta}) \tag{3.14}$$

and we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_+(t)} Q(t,x)\,dx \\
&\quad\leqslant \frac{c}{2}\int_{\Omega_+(t)} (1 + |u_m(t,x)|^{q/2})(1 + |u_m(t,x)|^{q/2\beta})\,|u_m''(t,x)|\,dx \\
&\quad\quad - c_4 \int_{\Omega_+(t)} \{|u_m'(t,x)|^2 + |u_m''(t,x)|^2\}\,dx \\
&\quad\leqslant \frac{c}{2}\left\{\int_{\Omega_+(t)} (1 + |u_m(t,x)|^{q(1+1/\beta)})\,dx\right\}^{1/2} \left\{\int_{\Omega_+(t)} |u_m''(t,x)|^2\,dx\right\}^{1/2} \\
&\quad\quad - c_4 \int_{\Omega_+(t)} \{|u_m'(t,x)|^2 + |u_m''(t,x)|^2\}\,dx.
\end{aligned} \tag{3.15}$$

Therefore it follows

$$\int_{\Omega_+(t)} Q(t,x)\,dx \leqslant c\sqrt{E_m(t)} - c_4 \int_{\Omega_+(t)} \{|u_m'(t,x)|^2 + |u_m''(t,x)|^2\}\,dx \tag{3.16}$$

where c is independent from t and m.

Let us consider now the other terms in (3.12); one has

$$\frac{1}{2}\int_{\Omega_-^{(n)}(t)\setminus\Omega_-^{(n+1)}(t)} Q(t,x)\,dx \leqslant -\frac{c_4}{2^{n+2}}\int_{\Omega_-^{(n)}(t)\setminus\Omega_-^{(n+1)}(t)} \{|u_m'(t,x)|^2 + |u_m''(t,x)|^2\}\,dx \tag{3.17}$$

$$\frac{1}{2}\int_{\tilde\Omega(t)} Q(t,x)\,dx \leqslant -\frac{c_4}{2}\int_{\tilde\Omega(t)} \{|u_m'(t,x)|^2 + |u_m''(t,x)|^2\}\,dx. \tag{3.18}$$

Finally, we obtain

$$\tfrac{1}{2}E_m(t) \leqslant c\sqrt{E_m(t)} - c_4 \int_\Omega \{|u_m'(t,x)|^2 + |u_m''(t,x)|^2\}\,dx. \tag{3.19}$$

From Eq. (3.6) we get

$$u_m'''(t,x) + (Au_m')(t,x) = \frac{\partial h}{\partial u}(x, u_m(t,x))\, u_m'(tx) + \frac{\partial g}{\partial v}(x, u_m'(t,x))\, u_m''(t,x). \tag{3.20}$$

Multiplying by $u_m'(t)$, it follows

$$\frac{d}{dt}\langle u_m'(t), u_m''(t)\rangle + |A^{1/2}u_m'(t)|_{0,2}^2 = \left\langle \frac{\partial h}{\partial u}(\cdot, u_m(t))\, u_m'(t), u_m''(t)\right\rangle + |u_m''(t)|_{0,2}^2 + \left\langle \frac{\partial}{\partial t} g(u_m'(t)), u_m'(t)\right\rangle. \tag{3.21}$$

Then if we denote by $J(v) = \int_0^v g(w)\, dw$

$$\left\langle \frac{\partial}{\partial t} g(\cdot, u_m'(t)), u_m'(t)\right\rangle = \frac{d}{dt}\left\{\langle g(\cdot, u_m'(t)), u_m'(t)\rangle - \int_\Omega J(x, u_m'(t,x))\, dx\right\}. \tag{3.22}$$

In this way we get

$$\frac{d}{dt}\left[\int_\Omega J(x, u_m'(t,x)) - u_m'(t,x)\, g(x, u_m'(t,x))\}\, dx + |A^{1/2}u_m'(t)|_{0,2}^2\right] = \left\langle \frac{\partial h}{\partial u}(\cdot, u_m(t))\, u_m'(t), u_m'(t)\rangle + |u_m''(t)|_{0,2}^2 - \frac{d}{dt}\langle u_m'(t), u_m''(t)\right\rangle. \tag{3.23}$$

Let us consider now

$$E_{0,m}(t) = \int_\Omega \{J(x, u_m'(t,x)) - u_m'(t,x)\, g(x, u_m'(t,x))\}\, dx. \tag{3.24}$$

Since g is a monotone decreasing function $E_0(t) \geqslant 0$ for all $t \geqslant 0$. Moreover let

$$H_m(t) = c_4 E_{0,m}(t) + E_m(t)$$

and it follows

$$
\begin{aligned}
\frac{1}{2} H_m'(t) &\leqslant c\sqrt{E_m(t)} - c_4\{|u_m'(t)|^2 + |u_m''(t)|^2\} \\
&\quad + \frac{c_4}{2}|u_m''(t)|_{0,2}^2 - \frac{c_4}{2}|A^{1/2}u_m'(t)|_{0,2}^2 \\
&\quad - \frac{d}{dt}\langle u_m'(t), u_m''(t)\rangle \\
&\leqslant \mathrm{const}(\sqrt{E_m(t)}+1) - c_5 E_m(t) - \frac{d}{dt}\langle u_m'(t), u_m''(t)\rangle.
\end{aligned}
\tag{3.25}
$$

Now two cases are possible. In the former

$$
\mathrm{const}(\sqrt{E_m(t)}+1) - c_5 E_m(t) \leqslant 0. \tag{3.26}
$$

Then

$$
\tfrac{1}{2}H_m(t) + \langle u_m'(t), u_m''(t)\rangle \leqslant \tfrac{1}{2}H_m(0) + \langle u_m'(0), u_m''(0)\rangle \leqslant K
$$

where K is independent from m.

Thus

$$
\begin{aligned}
\tfrac{1}{2}H_m(t) &\leqslant K + \mathrm{const}\,|u_m''(t)|_{0,2} \leqslant \mathrm{const}(1+\sqrt{E_m(t)}) \\
&\leqslant \mathrm{const}(1+\sqrt{H_m(t)}).
\end{aligned}
$$

So $H_m(t)$ is bounded by a constant independent from m. In the latter case

$$
c_5 E_m(t) \leqslant \mathrm{const}(\sqrt{E_m(t)}+1). \tag{3.27}
$$

Then $E_m(t)$ is bounded by a constant independent from m. In both cases we found a constant $C > 0$ independent from m such that

$$
|u_m''(t)|_{0,2} \leqslant C, \qquad |A^{1/2}u_m'(t)|_{0,2} \leqslant C
$$

for all $t \geqslant 0$.

Moreover

$$
|Au_m(t)|_{0,2} \leqslant |u_m''(t)|_{0,2} + |g(\cdot, u_m'(t))|_{0,2} + |h(\cdot, u_m(t))|_{0,2} \leqslant \mathrm{const}.
$$

In the case $N < 2m$ the sup norm is dominated by the $D_{1/2}$ norm, therefore it is possible to change the hypotheses of the above proposition in the following way.

PROPOSITION (3.2). *Assume that f is continuously differentiable and verifies the hypotheses of Theorem* (2.7). *If in addition*

$$\begin{gathered} f(x,u,v)=h(x,u)+g(x,v), \qquad x\in\bar{\Omega},\ u,v\in\mathbb{R} \\ \frac{\partial h}{\partial u}\leqslant 0, \qquad \frac{\partial g}{\partial v}\leqslant -b<0 \end{gathered} \tag{3.28}$$

then the Galerkin approximations verify the estimates

$$|u_m''(t)|_{0,2}\leqslant k, \qquad |A^{1/2}u_m'(t)|_{0,2}\leqslant k, \qquad |Au_m(t)|\leqslant k \tag{3.5$'$}$$

where $k>0$ is independent from m and t.

Proof. Denote by

$$\begin{aligned} E_m(t)&=|u_m''(t)|_{0,2}^2+|A^{1/2}u_m'(t)|_{0,2}^2 \\ H_m(t)&=\int_\Omega J(x,u_m'(t,x))\,dx-\langle g(\cdot,u_m'(t)),u_m'(t)\rangle \end{aligned} \tag{3.29}$$

where $J(v)=\int_0^v g(w)\,dw$.

Then one has

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}E_m(t)&=\left\langle\frac{\partial h}{\partial u}(\cdot,u_m(t))\,u_m'(t),u_m''(t)\right\rangle \\ &\quad+\left\langle\frac{\partial g}{\partial v}(\cdot,u_m'(t))\,u_m''(t),u_m''(t)\right\rangle \end{aligned} \tag{3.30}$$

$$\frac{d}{dt}H_m(t)=|u_m''(t)|^2-|A^{1/2}u_m'(t)|^2+\left\langle\frac{\partial h}{\partial u}(\cdot,u_m(t))\,u_m'(t),u_m'(t)\right\rangle. \tag{3.31}$$

In this way we get for all $\varepsilon>0$

$$\begin{aligned} \frac{d}{dt}\{E_m(t)+2H_m(t)\}&\leqslant\alpha\sqrt{E_m(t)}-2b\,|u_m''(t)|_{0,2}^2 \\ &\quad+\varepsilon\,|u_m''(t)|^2-\varepsilon\,|A^{1/2}u_m'(t)|^2 \end{aligned} \tag{3.32}$$

where

$$\alpha=2\sup\left\{\left|\frac{\partial h}{\partial u}(x,u)\right|:x\in\bar{\Omega},|u|\leqslant C\right\} \tag{3.33}$$

and C is the constant given in Proposition (2.8). If $\varepsilon=b$ it follows

$$\frac{d}{dt}\{E_m(t)+bH_m(t)\}\leqslant\alpha\sqrt{E_m(t)}-bE_m(t). \tag{3.34}$$

Using (3.34) we wish to prove the boundedness of $E_m + bH_m$ independently from m and t.

Indeed there exists $c_0 > 0$ independent from m and t such that

$$H_m(t) \geqslant \langle u''_m(t),\, u'_m(t)\rangle \leqslant -c_0 \sqrt{E_m(t)}. \tag{3.35}$$

Hence we get

$$E_m(t) + bH_m(t) \geqslant E_m(t) - c_0 \sqrt{E_m(t)}. \tag{3.36}$$

If for some $\tau > 0$ one has

$$E_m(\tau) + bH_m(\tau) < 0$$

then

$$\sqrt{E_m(\tau)} \leqslant c_0.$$

For this reason it follows, for all $t \geqslant 0$,

$$E_m(t) + bH_m(t) \geqslant -c_0^2. \tag{3.37}$$

Assume there exists $\rho > 0$ such that

$$\frac{d}{dt}\,\{E_m(\rho) + bH_m(\rho)\} \geqslant 0 \tag{3.38}$$

then by (3.34) one has

$$\sqrt{E_m(\rho)} \leqslant b/\alpha. \tag{3.39}$$

Therefore

$$\sup\{|u'_m(\rho, x)| : x \in \bar{\Omega}\} \leqslant c_1\, |A^{1/2} u'_m(\rho)|_{0,2} \leqslant c_1(b/\alpha) \tag{3.40}$$

and we can find a positive constant c_2 independent from m and such that

$$\begin{aligned} |H_m(\rho)| \leqslant \sup\{|\Omega|\, (J(x, v) - vg(x, v)) : |v| \leqslant c_1(b/\alpha), x \in \bar{\Omega}\} \\ + c_1(b/\alpha) \sqrt{E_m(\rho)} \leqslant c_2. \end{aligned} \tag{3.41}$$

Then for all $t \geqslant 0$

$$|E_m(t) + bH_m(t)| \leqslant \sup\{b^2/\alpha^2 + bc_2, E_m(0) + bH_m(0), c_0^2\}. \tag{3.42}$$

Since $\varphi \in D(A)$, $\psi \in D_{1/2}$ we can choose a basis $\{e_j\}$ so that

$$\begin{aligned} Au_m(0) \to A\varphi \qquad &\text{in} \quad L^2(\Omega) \\ A^{1/2}u'_m(0) \to A^{1/2}\psi \qquad &\text{in} \quad L^2(\Omega) \end{aligned} \tag{3.43}$$

as $m \to \infty$. Then $E_m(0)$ and $H_m(0)$ are bounded independently from m and we can find a constant $K_0 > 0$ so that, for all $t \geqslant 0$ and $m \in N$,

$$|E_m(t) + bH_m(t)| \leqslant k_0, \qquad |E_m(t)| \leqslant k_0. \tag{3.44}$$

Moreover let

$$\begin{aligned} M_1 &= \sup\{|h(x,u)| : x \in \bar{\Omega}, |u| \leqslant C\} \\ M_2 &= \sup\{|g(x,v)| : x \in \bar{\Omega}, |v| \leqslant c_1 \sqrt{k_0}\} \end{aligned} \tag{3.45}$$

and it follows

$$|Au_m(t)|_{0,2}^2 \leqslant |u_m''(t)|_{0,2}^2 + M_1^2 |\Omega| + M_2^2 |\Omega|. \tag{3.46}$$

We shall prove now that the precompactness of the orbits follows from our estimates on the Galerkin approximations and from the existence and uniqueness of the solution (see also the Appendix).

PROPOSITION (3.3). *Assume* (H) *has a unique classical solution such that, for all* $(\varphi, \psi) \in D(Q)$ *and for all* $T > 0$,

$$\begin{aligned} u &\in H^2(0,T,L^2(\Omega)) \cap C^1(0,T;D_{1/2}) \cap C(0,T;D(A)) \\ u' &\in H^1(0,T;L^2(\Omega)) \cap C(0,T;D_{1/2}) \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} u_m &\to u \qquad \text{in} \quad L^\infty(0,T;D_{1/2}) \\ u_m' &\to u' \qquad \text{in} \quad L^\infty(0,T;L^2(\Omega)). \end{aligned}$$

Moreover there exists $K > 0$ *such that*

$$|Au(t)|_{0,2} \leqslant K, \qquad |A^{1/2}u'(t)|_{0,2} \leqslant K, \qquad |u''(t)|_{0,2} \leqslant K$$

for all $t \geqslant 0$. *Then*

$$\gamma^+(\phi, \psi) = \{(u(t), u'(t)) \in X : t \geqslant 0\}$$

is precompact in X.

Proof. There exists a subsequence u_{m_j} of u_m such that

$$\begin{aligned} Au_{m_j} &\rightharpoonup Au \qquad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \text{ weak star} \\ A^{1/2}u_{m_j} &\rightharpoonup A^{1/2}u \qquad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \text{ weak star.} \end{aligned}$$

Indeed $L^\infty(0,T;L^2(\Omega))$ is the dual of $L^1(0,T;L^2(\Omega))$ and A is a closed operator. Then by the semicontinuity of the norm of $L^\infty(0, T; L^2(\Omega))$ with respect to the weak star topology

$$\operatorname*{ess\,sup}_{t\in[0,T]} |Au(t)|_{0,2} \leqslant \liminf_{j\to\infty}(\operatorname*{ess\,sup}_{t\in[0,T]} |Au_{m_j}(t)|_{0,2}) \leqslant k$$

$$\operatorname*{ess\,sup}_{t\in[0,T]} |A^{1/2}u'(t)|_{0,2} \leqslant \liminf_{j\to\infty}(\operatorname*{ess\,sup}_{t\in[0,T]} |A^{1/2}u'_{m_j}(t)|_{0,2}) \leqslant k.$$

4. Decay Estimates

Let us assume that for all initial data $(\phi, \psi) \in D(Q)$ there exists a unique solution

$$(u, u_t) \in C(0, T; D(Q)) \cap C^1(0, T; X)$$

and a constant $K > 0$ depending from (ϕ, ψ) such that the "a priori" bounds of Proposition (3.3) hold.

Proposition (4.1). *Assume f verify the following hypotheses:*

(i) *for all $u \in \mathbb{R}$, $x \in \Omega$*

$$f(x, u, 0)u \leqslant 2\int_0^u f(x, \sigma, 0)\, d\sigma \leqslant 0 \tag{4.1}$$

(ii) *for all $u, v \in \mathbb{R}$, $v \neq 0$, $x \in \Omega$*

$$-d(1 + |v|^{p-2}) \leqslant v^{-1}\{f(x, u, v) - f(x, u, 0)\} \leqslant -b\,|v|^{p-2} \tag{4.2}$$

where $b, d > 0$.

Therefore

$$|u_t(t)|_{0,2}^2 = O(t^{-1/p}), \qquad |A^{1/2}u(t)|_{0,2}^2 = O(t^{-1/p}) \qquad \text{as } t \to +\infty. \tag{4.3}$$

Proof. Denote by

$$E(t) = V(u(t), u_t(t)) \tag{4.4}$$

where V is the Liapunov functional given in (2.4). Therefore

$$\int_0^{+\infty} |u_t(s)|_{0,p}^p\, ds \leqslant b^{-1}E(0). \tag{4.5}$$

Let us define

$$E_1(t) = \int_\Omega u(t, x)\, u_t(t, x)\, dx \tag{4.6}$$

then

$$E_1'(t) \leqslant |u_t(t)|_{0,2}^2 - |A^{1/2}u(t)|_{0,2}^2 + \langle f(\cdot, u(t), 0), u(t)\rangle + \langle f(\cdot, u(t), u_t(t)) - f(\cdot, u(t), 0), u(t)\rangle.$$

From hypotheses (4.1) and (4.2) one has

$$\begin{aligned} E_1'(t) &\leqslant |u_t(t)|_{0,2}^2 - |A^{1/2}u(t)|_{0,2}^2 + 2\Phi(u(t)) \\ &\quad + \langle f(\cdot, u(t), u_t(t)) - f(\cdot, u(t), 0), u(t)\rangle \\ &\leqslant 2\,|u_t(t)|_{0,2}^2 - E(t) \\ &\quad + d\int_\Omega \{|u_t(t,x)|\,|u(t,x)| + |u_t(t,x)|^{p-1}\,|u(t,x)|\}\,dx. \end{aligned} \tag{4.7}$$

Using the monotonicity of $E(t)$ and (4.7)

$$\begin{aligned} E(t)t \leqslant \int_0^t E(s)\,ds &\leqslant (E_1(0) - E_1(t)) + 2\int_0^t |u(s)|_{0,2}^2\,ds \\ &\quad + d\int_0^t \int_\Omega \{|u_t(s,x)|\,|u(s,x)| + |u_t(s,x)|^{p-1}\,|u(s,x)|\}\,dx\,ds. \end{aligned} \tag{4.8}$$

Now there exists $C_0 > 0$ such that

$$E_1(0) - E_1(t) \leqslant |\phi|_{0,2}\,|\psi|_{0,2} + |u(t)|_{0,2}\,|u_t(t)|_{0,2} \leqslant C_0 \tag{4.9}$$

and let $Q_t = \Omega \times (0,t)$ and we can find $C_1 > 0$ such that

$$\begin{aligned} \int_0^t |u_t(s)|_{0,2}^2\,ds &= \iint_{Q_t} |u_t(s,x)|\,|u_t(s,x)|\,dx\,ds \\ &\leqslant \left[\iint_{Q_t} |u_t(s,x)|^p\,dx\,ds\right]^{1/p} \left[\iint_{Q_t} |u_t(s,x)|^{p'}\,dx\,ds\right]^{1/p'} \\ &\leqslant b^{-1/p'}E(0)^{1/p}\left[\int_0^t |u_t(s)|_{0,p'}^{p'}\,ds\right]^{1/p'} \\ &\leqslant b^{-1/p'}E(0)^{1/p}\,|\Omega|^{1-2/p'}\,E(0)^{p'/2}\,t^{1/p'}. \end{aligned} \tag{4.10}$$

Moreover there exists C_2, C_3, C_4, $C_5 > 0$ such that

$$\begin{aligned} &\iint_{Q_t} |u(s,x)|\,|u_t(s,x)|\,dx\,ds \\ &\quad\leqslant \int_0^t |u_t(s)|_{0,2}\,|u(s)|_{0,2}\,ds \\ &\quad\leqslant C_2 t^{-1/2p'}(\sup_{0\leqslant s\leqslant t} |A^{1/2}u(s)|_{0,2})\,t^{1/2p} \leqslant C_3 t^{-1/p'}t^{1/p} \end{aligned} \tag{4.11}$$

$$\begin{aligned}
&\iint_{Q_t} |u_t(s,x)|^{p-1}\,|u(s,x)|\,dx\,ds \\
&\quad\leqslant \left[\iint_{Q_t} |u_t(s,x)|^p\,dx\,ds\right]^{1/p'} \left[\iint_{Q_t} |u(s,x)|^p\,dx\,ds\right]^{1/p} \qquad (4.12)\\
&\quad\leqslant C_4 b^{-1/p'} E(0)^{1/p'}\,E(0)^{p/2}\,t^{1/p} \leqslant C_5 t^{1/p'}.
\end{aligned}$$

Therefore by (4.9)–(4.12) and by (4.8) one has

$$E(t)t \leqslant C_0 + C_1 t^{1/p'} + C_3 t^{1/p} t^{-1/p'} + C_5 t^{1/p'}.$$

Since $1/p' \leqslant \frac{1}{2} \leqslant 1/p$ one has

$$E(t)t \leqslant C_6 + C_7 t^{1/p'}.$$

Therefore $E(t) = O(t^{-1/p})$ as $t \to +\infty$.

Appendix

We wish to prove the existence and the uniqueness of the classical solutions (H), under the hypotheses of Propositions (3.1) and (3.2).

The first result is concerned with the existence of a solution.

Proposition (A.1). *Under the hypotheses of Propositions* (3.1) *and* (3.2), *if we assume the initial datum in* $D(Q)$ *then for all* $T > 0$ *there exists a solution*

$$\begin{aligned}
&u \in H^2(0,T; L^2(\Omega)) \cap C^1(0,T; D_{1/2}) \cap C(0,T; D(A)) \qquad (\text{A}.1)\\
&u' \in H^1(0,T; L^2(\Omega)) \cap C(0,T; D_{1/2})
\end{aligned}$$

such that

$$\begin{aligned}
&u_m \to u \qquad \text{in} \quad L^\infty(0,T; D_{1/2}) \\
&u'_m \to u' \qquad \text{in} \quad L^\infty(0,T; L^2(\Omega)).
\end{aligned} \qquad (\text{A}.2)$$

Proof. Let us denote by

$$\|u\|_{T,2} = \operatorname{ess\,sup}\{|u(t)|_{0,2} : t \in [0,T]\}$$

the norm of $L^\infty(0,T; L^2(\Omega))$ and by

$$\|u\|_{T,1/2} = \operatorname{ess\,sup}\{|A^{1/2}u(t)|_{0,2} : t \in [0,T]\}$$

the norm of $L^\infty(0,T; D_{1/2})$.

By the estimate $\|Au\|_{T,2} \leqslant K$ we obtain a subsequence denoted $\{u_m\}$ again, such that

$$\begin{aligned} Au_m &\rightharpoonup Au && \text{in} \quad L^\infty(0,T;L^2(\Omega)) \text{ weak star} \\ A^{1/2}u_m' &\rightharpoonup A^{1/2}u' && \text{in} \quad L^\infty(0,T;L^2(\Omega)) \text{ weak star} \end{aligned} \tag{A.3}$$

where $u \in L^\infty(0, T; D_{1/2})$.

Moreover

$$\begin{aligned} &\operatorname*{ess\,sup}_{t\in[0,T]} \int_\Omega |h(x, u_m(t,x)) - h(x, u(t,x))|\,dx \\ &\quad\leqslant \operatorname*{ess\,sup}_{t\in[0,T]} \int_\Omega \int_0^1 \left|\frac{\partial h}{\partial u}(x, u_m(t,x)) + \lambda(u(t,x) - u_m(t,x))\right| \\ &\quad\quad \times |u_m(t,x) - u(t,x)|\,d\lambda\,dx \\ &\quad\leqslant c_2 \int_\Omega (1 + |u_m(t,x)|^{p/2-1} + |u(t,x) - u_m(t,x)|^{p/2-1}) \\ &\quad\quad \times |u_m(t,x) - u(t,x)|\,dx \\ &\quad\leqslant \text{const} \left\{\int_\Omega |u_m(t,x) - u(t,x)|\,dx + \int_\Omega |u_m(t,x)|^{(p-2)/2} \right. \\ &\quad\quad \left. \times |u_m(t,x) - u(t,x)|\,dx + \int_\Omega |u_m(t,x) - u(t,x)|^{p/2}\,dx\right\} \\ &\quad\leqslant \text{const}\,\|u - u_m\|_{T,1/2}. \end{aligned} \tag{A.4}$$

Therefore there exists a subsequence of $\{u_m\}$ such that

$$h(x, u_m(t,x)) \to h(x, u(t,x)) \qquad \text{a.e.} \quad (t,x) \in (0,T) \times \Omega. \tag{A.5}$$

Moreover there exists $K_1 > 0$ such that

$$\|h(u_m)\|_{T,2} \leqslant k_1. \tag{A.6}$$

Then by Lions [29, p. 12, Lemma (1.3)] it follows

$$h(u_m) \rightharpoonup h(u) \qquad \text{in} \quad L^2(0,T;L^2(\Omega)) \text{ weak.} \tag{A.7}$$

Let us consider now the other estimates.

By $\|A^{1/2}u_m'\| \leqslant k$ we get a subsequence of $\{u_m'\}$ such that

$$\begin{aligned} A^{1/2}u_m' &\rightharpoonup A^{1/2}v && \text{in} \quad L^\infty(0,T;L^2(\Omega)) \text{ weak star} \\ u_m' &\to v && \text{in} \quad L^\infty(0,T;L^2(\Omega)) \end{aligned} \tag{A.8}$$

where $v \in L^\infty(0, T; L^2(\Omega))$.

Since d/dt is a closed operator on $L^\infty(0, T; L^2(\Omega))$ it follows

$$v = u' \qquad \text{a.e.} \quad \text{in} \quad (t, x) \in (0, T) \times \Omega$$

and

$$\begin{aligned} A^{1/2} u'_m &\rightharpoonup A^{1/2} u' \qquad &&\text{in} \quad L^\infty(0, T; L^2(\Omega)) \text{ weak star} \\ u'_m &\to u' &&\text{in} \quad L^\infty(0, T; L^2(\Omega)). \end{aligned} \tag{A.9}$$

Finally, since $\|u''_m\|_{T,2} \leqslant K$ one has

$$u''_m \rightharpoonup u'' \qquad \text{in} \quad L^\infty(0, T; L^2(\Omega)) \text{ weak star.}$$

From

$$u' \in L^\infty(0, T; D_{1/2}), \qquad u'' \in L^\infty(0, T; L^2(\Omega))$$

it follows

$$u' \in C(0, T; D_{1/2}), \qquad u \in C^1(0, T; D_{1/2}) \cap C(0, T; D(A)). \tag{A.10}$$

Moreover $Au \in C(0, T; L^2(\Omega))$.

Choose $r > 0$ sufficiently small, then it follows

$$\int_\Omega |g(x, u'_m(t, x)) - g(x, u'(t, x))|^r \, dx \leqslant \text{const} \, \|u - u_m\|_{T,2}$$

and

$$\|g(\cdot, u_m(\cdot))\|_{T,2} \leqslant \text{const}.$$

Then there exists a subsequence such that

$$g(\cdot, u_m(\cdot)) \rightharpoonup g(\cdot, u(\cdot)) \qquad \text{in} \quad L^2(0, T; L^2(\Omega)) \text{ weak.}$$

Hence

$$u'' + Au = h(u) + g(u').$$

The next result is concerned with the uniqueness of the solution. Observe that the solution we found is the limit of the Galerkin approximations, and for this reason we got good estimates on the higher-order derivatives, but in general these estimates are not true for whatever a solution. The next uniqueness result say that all the classical solutions are limits of finite-dimensional approximations so they have such good estimate.

PROPOSITION (A.2). *Let u the above solution and*

$$v \in H^2(0, T; L^2(\Omega)) \cap C^1(0, T; D_{1/2}) \cap C(0, T, D(A))$$
$$v' \in H^1(0, T; L^2(\Omega)) \cap C(0, T; D_{1/2})$$

another solution to (H) *having the same initial datum* $(\varphi, \psi) \in D(Q)$. *Then for all* $t \in [0, T]$

$$u(t, x) = v(t, x) \qquad \text{a.e. in } x \in \Omega.$$

Proof. Let us denote by $z = u - v$. Then z verifies the equation.

$$\begin{gathered} z''(t) + Az(t) = h(\cdot, u(t)) - h(\cdot, v(t)) + g(\cdot, u'(t)) - g(\cdot, v'(t)) \\ z(0) = 0, \qquad z'(0) = 0. \end{gathered} \tag{A.11}$$

Multiplying by z' we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \{|z'(t)|_{0,2}^2 + |A^{1/2} z(t)|_{0,2}^2\} \\ &\qquad \leqslant \text{const}\{|z'(t)|_{0,2} + |u(t)|_{0,p}^{(p-2/2} |z(t)|_{0,p} + |z(t)|_{0,p}^{p/2} |z'(t)|_{0,2}\}. \end{aligned}$$

Let $e(t) = \frac{1}{2}(|A^{1/2} z(t)|_{0,2}^2 + |z'(t)|_{0,2}^2)$, then

$$\begin{gathered} e'(t) \leqslant \text{const}(e(t) + e(t)^{p/2}) \\ e(0) = 0,\ e(t) \geqslant 0. \end{gathered} \tag{A.12}$$

By the theory of differential inequality $e(t) = 0$, for all $t \geqslant 0$.

Remark (A.3). The hypothesis (3.4) in Prop. (3.1) can be weakened in he following way. Let us assume $q = p - 2$, $\beta = (p/2) - 1$ and the existence of two positive constants c_3, c_4 such that

$$\begin{gathered} -c_3(1 + |v|^{(p/2)-1}) \leqslant v^{-1} g(x, v) \leqslant -c_4 |v|^{(p/2)-1}, \qquad v \neq 0 \\ \frac{\partial g}{\partial v}(x, v) \leqslant -c_4 |v|^{(p/2)-1}. \end{gathered} \tag{A.13}$$

Then the orbits are precompact also in this case.

Indeed let us consider the approximating equation

$$u_\varepsilon'' + Au_\varepsilon = h(u_\varepsilon) + g(u_\varepsilon') - \varepsilon u_\varepsilon', \qquad u_\varepsilon(0) = u(0),\ u_\varepsilon'(0) = u'(0) \tag{A.14}$$

for all $\varepsilon > 0$.

It is easy to see that

$$|u'_\varepsilon(t)|^2_{0,2} + |A^{1/2}u_\varepsilon(t)|^2_{0,2} \tag{A.15}$$

is boundended uniformly in t and ε.

Moreover the Proposition (3.1) and the existence theorem (A.2) can be applied to the approximating equation (A.14) in order to get a smooth solutions net $\{(u_\varepsilon, u'_\varepsilon)\}_{\varepsilon>0}$.

By formula (3.25) applied to these perturbed problems we find a positive constant $c > 0$ such that

$$\frac{1}{2}\dot{H}_\varepsilon(t) + c_4 \frac{d}{dt}\langle u'_\varepsilon, u''_\varepsilon\rangle \leqslant c(\sqrt{E_\varepsilon(t)} - \varepsilon E_\varepsilon(t)). \tag{A.16}$$

Hence it follows

$$\tfrac{1}{2}H_\varepsilon(t) + c_4\langle u'_\varepsilon(t), u''_\varepsilon(t)\rangle \leqslant \tfrac{1}{2}H(0) + c_4\langle u'(0), u''_\varepsilon(0)\rangle + c\int_0^t \sqrt{E_\varepsilon(s)}\,ds. \tag{A.17}$$

Therefore there exist $a, b > 0$ independent of ε, such that

$$E_\varepsilon(t) \leqslant a\int_0^t \sqrt{E_\varepsilon(s)}\,ds + b. \tag{A.18}$$

Then for all $T > 0$ there exists $c(T) > 0$, independent of ε, such that

$$\sup\{E_\varepsilon(t) \colon t \in [0, T]\} \leqslant c(T). \tag{A.19}$$

If we denote by (u_m, u'_m) the Galerkin approximations for the unperturbed problem one has

$$|A^{1/2}u_m(t)|^2_{0,2} + |u''_m(t)|^2_{0,2} \leqslant c(T) \tag{A.20}$$

for all $T > 0$ and $t \in [0, T]$.

These "a priori" bounds on each compact interval $[0, T]$ are sufficient to repeat the computations of Propositions (A.2) and (A.3) providing us the existence and the uniqueness of a smooth solution to our problem. Since the above estimates are not uniform in T, in general, they do not give any precompactness properties of the orbits.

However, we will be able to obtain the precompactness by proving directly the convergence towards a suitable limit of each bounded sequence contained in the orbit. (Actually we will prove that each one of these sequences tends to zero.)

Arguing by contradiction, we assume that

$$V(t) = \tfrac{1}{2}\{|u'(t)|_{0,2}^2 + |A^{1/2}u(t)|_{0,2}^2 - 2\Phi(u(t))\} \tag{A.21}$$

does not converge to zero as t goes to infinity.

Therefore, since $\dot{V}(t) \leqslant 0$, it follows there exists $\sigma > 0$ such that $V(t) \geqslant \sigma$, for all $t \geqslant 0$. Let us denote by

$$F(t) = V(t) + \gamma\left[\langle u(t), u'(t)\rangle - \int_0^t \langle g(u'(s)), u(s)\rangle\, ds\right], \qquad \gamma > 0 \tag{A.22}$$

then one has

$$\begin{aligned}\dot{F}(t) &\leqslant -c_5\,|u'(t)|_{0,2}^{(p/2)+1} + \gamma[|u'(t)|_{0,2}^2 - |A^{1/2}u(t)|_{0,2}^2],\\ c_5 &= c_4\,|\Omega|^{[(p/2)-1]/(p+2)}.\end{aligned} \tag{A.23}$$

By standard calculus arguments we can prove that if $z^2 + y^2 \geqslant 2\sigma$ then there exist $\gamma(\sigma) > 0$ and $c(\sigma) > 0$ such that

$$-c_5 z^{(p/2)+1} + \gamma(\sigma)[z^2 - y^2] \leqslant -c(\sigma) < 0 \tag{A.24}$$

and hence it follows

$$F(t) \leqslant F(0) - c(0)t \tag{A.25}$$

for all $t \geqslant 0$. Moreover, since $\dot{V}(t) = -\langle g(u'), u'\rangle$, there exists $d_0 > 0$ such that

$$d_0 \int_0^{+\infty} |u'(t)|_{0,2}^{(p/2)+1}\, dt \leqslant c_4 \int_0^{\infty} |u'(t)|_{0,(p/2)+1}^{(p/2)+1}\, dt \leqslant V(0). \tag{A.26}$$

Using this inequality we can find two positive constants d_1, d_2 such that

$$|F(t)| \leqslant d_1 + d_2 t^{p/(p+2)}. \tag{A.27}$$

Thus it follows

$$|F(t)| = o(t) \tag{A.28}$$

as t tends to infinity. Then by (A.25) we obtain

$$-|o(t)| \leqslant F(t) \leqslant F(0) - c(\sigma)t$$

which is clearly false.

EXAMPLES

(A) Let Ω be an open bounded subset of $\mathbb{R}^3$; we wish to study the equation

$$\begin{aligned} &u_{tt} - \Delta u + u^3 + u_t^3 = 0, && u_t = \partial u/\partial t \\ &u(0,x) = \varphi(x), && u_t(0,x) = \psi(x) \\ &u(t,x) = 0, && x \in \partial\Omega,\ t \geqslant 0. \end{aligned} \tag{E.1}$$

Then $N = 3$, $m = 1$, so that $2m > N$. In order to apply Proposition (3.1) to this example we choose $q = 4$, $\beta = 2$ and $p = 6 = 2N/(N-2m)$. Then for all initial data $(\varphi, \psi) \in H^2(\Omega) \cap H_0^1(\Omega) \oplus H_0^1(\Omega)$ there exists a unique classical solution to (E.1) and moreover

$$\lim_{t\to+\infty} |Du(t)|_{0,2} = 0, \qquad \lim_{t\to+\infty} |u'(t)|_{0,2} = 0, \qquad D = \text{grad}.$$

(B) Let Ω be a bounded open subset of $\mathbb{R}^4$, and let us consider the equation

$$\begin{aligned} &u_{tt} + \Delta^2 u + u^5 + u_t^7 = 0 \\ &u(0,x) = \varphi(x), \qquad u_t(0,x) = \psi(x) \\ &\left.\begin{aligned} &u(t,x) = 0 \\ &\frac{\partial u}{\partial n}(t,x) = 0 \end{aligned}\right\} \quad \text{if} \quad x \in \partial\Omega,\ t \geqslant 0. \end{aligned} \tag{E.2}$$

In this case $N = 4$ and $m = 2$ then $2m = N$. So that $2N/(N-2m) = +\infty$ and p can be chosen in the interval $[2, +\infty)$.

Therefore $h(u) = -u^5$, $g(v) = -v^7$ imply $q = 4$ and $\beta = 6$. From the condition $q(1 + 1/\beta) \leqslant p$ it follows $p \geqslant 14/3$. Moreover since we want

$$|h(u)| \leqslant c(1 + |u|^{p/2})$$

we need $p \geqslant 10$. But it is required also

$$|g(v)| \leqslant c(1 + |v|^{p/2})$$

then $p \geqslant 14$. Hence by Proposition (3.1) one obtains the a priori estimates that ensure the existence of a unique classical solution for a smooth initial datum. In addition for any $\sigma \geqslant 2$

$$\lim_{t\to+\infty} |\Delta u(t)|_{0,2} = \lim_{t\to+\infty} |u(t)|_{0,\sigma} = 0$$

$$\lim_{t\to+\infty} |u'(t)|_{0,2} = 0.$$

(C) Let us consider now the following one-dimensional equation:

$$\begin{gathered} u_{tt} + u_{xxxx} = -u \lg(|u| + 1) - u_t \exp(u_t^2) \\ u(0, x) = \varphi(x), \qquad u_t(0, x) = \psi(x), \qquad t \geqslant 0,\ x \in [0, \pi] \\ u(t, 0) = u(t, \pi) = 0 \\ u_x(t, 0) = u_x(t, \pi) = 0. \end{gathered} \tag{E.3}$$

we can apply here Proposition (3.2) and we obtain a classical solution such that for all $T > 0$, $u \in C^1(0, T;\ C[0, \pi])$. Moreover

$$\lim_{t \to +\infty} |u(t)|_{0,\infty} = 0, \qquad \lim_{t \to +\infty} |u_t(t)|_{0,2} = 0.$$

Concluding remarks. Using the theory developed in Ball–Slemrod [30, 31], it is possible to treat some cases studied in this paper. The author wishes to thank the referee for having suggested these references.

References

1. L. Amerio and G. Prouse, "Almost Periodic Functions and Functional Equations," Van Nostrand–Reinhold, New York, 1971.
2. R. A. Adams, "Sobolev Spaces," Academic Press, New York, 1975.
3. E. A. Barbashin and N.N. Krasovskii, Stability of the motion in the large *DAN* **86** (1952), 3.
4. H. Berestycki and P. L. Lions, Nonlinear scalar field equations, preprint.
5. M. Biroli and A. Haraux, Asymptotic behavior for an almost periodic strongly dissipative wave equation, *J. Differential Equations* **38** (1980), 422–440.
6. M. Biroli, Bounded or almost periodic solutions of the nonlinear vibrating membrane equation, *Ricerche Mat.* **22** (1973), 190–202.
7. H. Brezis, "Operateurs Maximaux monotones et semigroupes de contractions dans les espaces de Hilbert," North-Holland, Amsterdam/London, 1973.
8. T. Cazenave, Stable solutions of the logarithmic Schrödinger equation, preprint.
9. C. M. Dafermos, An invariance principle for compact processes, *J. Differential Equations* **9** (1971), 239–252.
10. C. M. Dafermos, Applications of the invariance principle for compact processes, *J. Differential Equations* **9** (1971), 291–299.
11. C. M. Dafermos and M. Slemrod, Asymptotic behavior of nonlinear contraction semigroups, *J. Differential Equations* **13** (1973), 97–106.
12. B. M. D'Onofrio, The stability problem for some nonlinear evolution equations, *Boll. Uni. Mat. Ital. B* **17** (1980), 425–439.
13. A. Friedman, "Partial Differential Equations," Holt, Rinehart & Winston, New York, 1969.
14. J. K. Hale, Dynamical systems and stability, *J. Math. Anal. Appl.* **26** (1969), 39–59.
15. A. Haraux, Comportment a l'infini pour certains systemes disspatifs nonlineaires, *Proc. R. Soc. Edinb.* **84A** (1979), 213–234.
16. A. Haraux, "Nonlinear Evolution Equations–Global Behavior of the Solutions," Lecture Notes in Mathematics No. 841, Springer-Verlag, Berlin/Heidelberg/New York, 1981.

17. D. HENRY, "Geometric theory of semilinear parabolic equations," Lecture Notes in Mathematics, No. 840, Springer-Verlag, Berlin, 1981.
18. M. A. KRASNOSELSKII, "Topological Methods in the Theory of Nonlinear Integral Equations," Macmillan Co., New York, 1964.
19. J. P. LA SALLE, "The Stability of Dynamical Systems," SIAM Regional Conference Series in Applied Mathematics **25**, 1976.
20. J. L. LIONS AND E. MAGENES, "Problèmes aux limites non homogènes et applications," Vols. I–II, Dunod, Paris, 1968.
21. M. NAKAO, Decay of solutions of same nonlinear evolution equation, *J. Math. Anal. Appl.* **60** (1977), 542–549.
22. M. NAKAO, Decay of solutions of some nonlinear wave equations in one space dimension, *Funkcial. Ekvac.* **29** (1977), 223–236.
23. T. NARAZAKI, Existence and decay of classical solutions of some nonlinear evolution equations with strong dissipation, *Proc. Fac. Sci. Tokai Univ.* **25** (1979), 45–62.
24. M. REED, Higher order estimates and smoothness of nonlinear wave equations, *Proc. Amer. Math. Soc.* **51** (1975), 79–85.
25. L. SALVADORI, Introduzione alla teoria della Stabilità del movimento, Lecture notes, University of Trento, 1979–1980.
26. I. SEGAL, Nonlinear semigroups, *Ann. of Math.* **78** (1963), 339–364.
27. M. YAMGUCHI, Existence and stability of global bounded classical solutions of initial boundary value problem for semilinear wave equations, *Funkcial. Ekvac.* **23** (1980), 289–308.
28. G. F. WEBB, A bifurcation problem for a nonlinear hyperbolic partial differential equation, *SIAM J. Math. Anal.* **10** (1979), 922–932.
29. J. L. LIONS, "Quelques méthodes de résolution des problèmes aux limites nonlinéaires," Dunod, Paris, 1969.
30. J. M. BALL AND M. SLEMROD, Feedback stabilization of distributed semilinear control systems, *Appl. Math. Optim.* **5** (1979), 169–179.
31. J. M. BALL AND M. SLEMROD, Nonharmonic Fourier series in the stabilization of distributed semilinear control systems, *Comm. Pure Appl. Math.* **32** (1979), 555–587.